q-ANALOGUE OF p-ADIC $\log\Gamma$ TYPE FUNCTIONS ASSOCIATED WITH MODIFIED q-EXTENSION OF GENOCCHI NUMBERS WITH WEIGHT α AND β

SERKAN ARACI AND MEHMET ACIKGOZ

Abstract. The fundamental aim of this paper is to describe q-Analogue of p-adic log gamma functions with weight alpha and beta. Moreover, we give relationship between p-adic q-log gamma funtions with weight (α,β) and q-extension of Genocchi numbers with weight alpha and beta and modified q-Euler numbers with weight α

1. Introduction

Assume that p be a fixed odd prime number. Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote by the ring of integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Also we denote $\mathbb{N}^*=\mathbb{N}\cup\{0\}$ and $\exp(x)=e^x$. Let $v_p:\mathbb{C}_p\to\mathbb{Q}\cup\{\infty\}$ ($\mathbb{Q}$ is the field of rational numbers) denote the p-adic valuation of $\mathbb{C}_p$ normalized so that $v_p(p)=1$. The absolute value on $\mathbb{C}_p$ will be denoted as $|.|_p$, and $|x|_p=p^{-v_p(x)}$ for $x\in\mathbb{C}_p$. When one talks of q-extensions, q is considered in many ways, e.g. as an indeterminate, a complex number $q\in\mathbb{C}$, or a p-adic number $q\in\mathbb{C}_p$, If $q\in\mathbb{C}$ we assume that $|q|<1$. If $q\in\mathbb{C}_p$, we assume $|1-q|_p<p^{-\frac{1}{p-1}}$, so that $q^x=\exp(x\log q)$ for $|x|_p\leq 1$. We use the following notation

$$[x]_q=\frac{1-q^x}{1-q},\quad [x]_{-q}=\frac{1-(-q)^x}{1+q} \tag{1.1}$$

where $\lim_{q\to 1}[x]_q=x$; cf. [1-24].

For a fixed positive integer d with $(d,f)=1$, we set

$$X = X_d=\varprojlim_{N}\mathbb{Z}/dp^N\mathbb{Z},$$
$$X^* = \underset{\substack{0<a<dp\\(a,p)=1}}{\cup} a+dp\mathbb{Z}_p$$

and

$$a+dp^N\mathbb{Z}_p=\left\{x\in X\mid x\equiv a\left(\operatorname{mod} dp^N\right)\right\},$$

where $a\in\mathbb{Z}$ satisfies the condition $0\leq a<dp^N$.

It is known that

$$\mu_q\left(x+p^N\mathbb{Z}_p\right)=\frac{q^x}{[p^N]_q}$$

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is a distribution on X for $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$.

Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable function on $\mathbb{Z}_p$. We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient

$$F_f(x,y) = \frac{f(x) - f(y)}{x-y}$$

has a limit $f^{\prime}(a)$ as $(x,y) \to (a,a)$ and denote this by $f \in UD(\mathbb{Z}_p)$. The p-adic q-integral of the function $f \in UD(\mathbb{Z}_p)$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)\, q^x \tag{1.2}$$

The bosonic integral is considered by Kim as the bosonic limit $q \to 1$, $I_1(f) = \lim_{q\to 1} I_q(f)$. Similarly, the p-adic fermionic integration on $\mathbb{Z}_p$ defined by Kim as follows:

$$I_{-q}(f) = \lim_{q\to -q} I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x)$$

Let $q \to 1$, then we have p-adic fermionic integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \lim_{q\to -1} I_q(f) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x .$$

Stirling asymptotic series are defined by

$$\log\left(\frac{\Gamma(x+1)}{\sqrt{2\pi}}\right) = \left(x - \frac{1}{2}\right)\log x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \frac{B_{n+1}}{x^n} - x \tag{1.3}$$

where B_n are familiar n-th Bernoulli numbers cf. [6, 8, 9, 25].

Recently, Araci et al. defined modified q-Genocchi numbers and polynomials with weight α and β in [4, 5] by the means of generating function:

$$\sum_{n=0}^{\infty} g_{n,q}^{(\alpha,\beta)}(x) \frac{t^n}{n!} = t \int_{\mathbb{Z}_p} q^{-\beta\xi} e^{[x+\xi]_{q^\alpha} t} d\mu_{-q^\beta}(\xi) \tag{1.4}$$

So from above, we easily get Witt's formula of modified q-Genocchi numbers and polynomials with weight α and β as follows:

$$\frac{g_{n+1,q}^{(\alpha,\beta)}(x)}{n+1} = \int_{\mathbb{Z}_p} q^{-\beta\xi} [x+\xi]_{q^\alpha}^n \, d\mu_{-q^\beta}(\xi) \tag{1.5}$$

where $g_{n,q}^{(\alpha,\beta)}(0) := g_{n,q}^{(\alpha,\beta)}$ are modified q extension of Genocchi numbers with weight α and β cf. [4,5].

In [21], Rim and Jeong are defined modified q-Euler numbers with weight α as follows:

$$\widetilde{\xi}_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p} q^{-t} [t]_{q^\alpha} \, d\mu_{-q}(t) \tag{1.6}$$

From expressions of (1.5) and (1.6), we get the following Proposition 1:

Proposition 1. *The following*

$$\widetilde{\xi}_{n,q}^{(\alpha)}=\frac{g_{n+1,q}^{(\alpha,1)}}{n+1} \tag{1.7}$$

is true.

In previous paper [6], Araci, Acikgoz and Park introduced weighted q-Analogue of p-Adic log gamma type functions and they derived some interesting identities in Analytic Numbers Theory and in p-Adic Analysis. They were motivated from paper of T. Kim by "*On a q-analogue of the p-adic log gamma functions and related integrals*, *J. Number Theory*, *76 (1999), no. 2, 320-329*." We also introduce q-Analogue of p-Adic log gamma type function with weight α and β. We derive in this paper some interesting identities this type of functions.

On p-adic $\log \Gamma$ **function with weight** α and β

In this part, from (1.2), we begin with the following nice identity:

$$I_{-q}^{(\beta)}\left(q^{-\beta x}f_n\right)+(-1)^{n-1}\,I_{-q}^{(\beta)}\left(q^{-\beta x}f\right)=[2]_{q^{\beta}}\sum_{l=0}^{n-1}(-1)^{n-1-l}\,f\left(l\right) \tag{1.8}$$

where $f_n(x)=f(x+n)$ and $n\in\mathbb{N}$ (see [4]).

In particular for $n=1$ into (1.8), we easily see that

$$I_{-q}^{(\beta)}\left(q^{-\beta x}f_1\right)+I_{-q}^{(\beta)}\left(q^{-\beta x}f\right)=[2]_{q^{\beta}}\,f\left(0\right). \tag{1.9}$$

With the simple application, it is easy to indicate as follows:

$$\left((1+x)\log(1+x)\right)^{\prime}=1+\log(1+x)=1+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,(n+1)}x^{n} \tag{1.10}$$

where $\left((1+x)\log(1+x)\right)^{\prime}=\frac{d}{dx}\left((1+x)\log(1+x)\right)$

By expression of (1.10), we can derive

$$(1+x)\log(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,(n+1)}x^{n+1}+x+c,\ \text{where } c \text{ is constant.} \tag{1.11}$$

If we take $x=0$, so we get $c=0$. By expression of (1.10) and (1.11), we easily see that,

$$(1+x)\log(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,(n+1)}x^{n+1}+x. \tag{1.12}$$

It is considered by T. Kim for q-analogue of p adic locally analytic function on $\mathbb{C}_p\backslash\mathbb{Z}_p$ as follows:

$$G_{p,q}(x)=\int_{\mathbb{Z}_p}[x+\xi]_q\left(\log[x+\xi]_q-1\right)d\mu_{-q}(\xi)\ \text{(for detail, see[5,6]).} \tag{1.13}$$

By the same motivation of (1.13), in previous paper [6], q-analogue of p-adic locally analytic function on $\mathbb{C}_p\backslash\mathbb{Z}_p$ with weight α is considered

$$G_{p,q}^{(\alpha)}(x)=\int_{\mathbb{Z}_p}[x+\xi]_{q^{\alpha}}\left(\log[x+\xi]_{q^{\alpha}}-1\right)d\mu_{-q}(\xi) \tag{1.14}$$

In particular $\alpha=1$ into (1.14), we easily see that, $G_{p,q}^{(1)}(x)=G_{p,q}(x)$.

With the same manner, we introduce q-Analoge of p-adic locally analytic function on $\mathbb{C}_p \backslash \mathbb{Z}_p$ with weight α and β as follows:

$$G_{p,q}^{(\alpha,\beta)}(x) = \int_{\mathbb{Z}_p} q^{-\beta\xi} [x+\xi]_{q^\alpha} \left(\log [x+\xi]_{q^\alpha} - 1\right) d\mu_{-q^\beta}(\xi) \tag{1.15}$$

From expressions of (1.9) and (1.16), we state the following Theorem:

Theorem 1. *The following identity holds:*

$$G_{p,q}^{(\alpha,\beta)}(x+1) + G_{p,q}^{(\alpha,\beta)}(x) = [2]_{q^\beta} [x]_{q^\alpha} \left(\log [x]_{q^\alpha} - 1\right).$$

It is easy to show that,

$$\begin{aligned} [x+\xi]_{q^\alpha} &= \frac{1-q^{\alpha(x+\xi)}}{1-q^\alpha} \\ &= \frac{1-q^{\alpha x}+q^{\alpha x}-q^{\alpha(x+\xi)}}{1-q^\alpha} \\ &= \left(\frac{1-q^{\alpha x}}{1-q^\alpha}\right) + q^{\alpha x}\left(\frac{1-q^{\alpha\xi}}{1-q^\alpha}\right) \\ &= [x]_{q^\alpha} + q^{\alpha x} [\xi]_{q^\alpha} \end{aligned} \tag{1.16}$$

Substituting $x \to \frac{q^{\alpha x}[\xi]_{q^\alpha}}{[x]_{q^\alpha}}$ into (1.12) and by using (1.16), we get interesting formula:

$$[x+\xi]_{q^\alpha}\left(\log [x+\xi]_{q^\alpha} - 1\right) = \left([x]_{q^\alpha} + q^{\alpha x}[\xi]_{q^\alpha}\right) \log [x]_{q^\alpha} + \sum_{n=1}^{\infty} \frac{(-q^{\alpha x})^{n+1}}{n(n+1)} \frac{[\xi]_{q^\alpha}^{n+1}}{[x]_{q^\alpha}^{n}} - [x]_{q^\alpha} \tag{1.17}$$

If we substitute $\alpha = 1$ into (1.17), we get Kim's q-Analogue of p-adic log gamma fuction (for detail, see[8]).

From expression of (1.2) and (1.17), we obtain worthwhile and interesting theorems as follows:

Theorem 2. *For $x \in \mathbb{C}_p \backslash \mathbb{Z}_p$ the following*

$$G_{p,q}^{(\alpha,\beta)}(x) = \left(\frac{[2]_{q^\beta}}{2}[x]_{q^\alpha} + q^{\alpha x}\frac{g_{2,q}^{(\alpha,\beta)}}{2}\right)\log [x]_{q^\alpha} + \sum_{n=1}^{\infty} \frac{(-q^{\alpha x})^{n+1}}{n(n+1)(n+2)} \frac{g_{n+1,q}^{(\alpha,\beta)}}{[x]_{q^\alpha}^{n}} - [x]_{q^\alpha}\frac{[2]_{q^\beta}}{2} \tag{1.18}$$

is true.

Corollary 1. *Taking $q \to 1$ into (1.18), we get nice identity:*

$$G_{p,1}^{(\alpha,\beta)}(x) = \left(x + \frac{G_2}{2}\right)\log x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)(n+2)} \frac{G_{n+1}}{x} - x$$

where G_n are called famous Genocchi numbers.

Theorem 3. *The following nice identity*

$$G_{p,q}^{(\alpha,1)}(x) = \left(\frac{[2]_q}{2}[x]_{q^\alpha} + q^{\alpha x}\widetilde{\xi}_{1,q}^{(\alpha)}\right)\log [x]_{q^\alpha} + \sum_{n=1}^{\infty} \frac{(-q^{\alpha x})^{n+1}}{n(n+1)} \frac{\widetilde{\xi}_{n,q}^{(\alpha)}}{[x]_{q^\alpha}^{n}} - \frac{[2]_q}{2}[x]_{q^\alpha} \tag{1.19}$$

is true.

Corollary 2. *Putting $q \to 1$ into (1.19), we have the following identity:*

$$G_{p,1}^{(\alpha,\beta)}(x) = (x + E_1)\log x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \frac{E_n}{x^n} - x$$

where E_n are familiar Euler numbers.

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University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY
E-mail address: mtsrkn@hotmail.com

University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY
E-mail address: acikgoz@gantep.edu.tr